The Anatomy of Principal Protected Absolute Return Barrier Notes

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Abstract

Principal Protected Absolute Return Barrier Notes (ARBNs) are structured products that guarantee to return the face value of the note at maturity and pay interest if the underlying security's price does not vary excessively. In this paper we derive four closed-form valuation approaches which are considered as representative methodologies on valuing structured products. Our approaches are: 1) decomposing an ARBN's payoff into double-barrier linear segment options, 2) decomposing an ARBN's payoff into double-barrier call and put options, 3) transforming an ARBN's path-dependent payoff rule into a path-independent payoff rule which significantly simplifies the derivation of product value, and 4) using PDE (Partial Differential Equations) to model an ARBN's payoff and calculate its value. We use the four methodologies to value 214 publicly-listed ARBNs issued by six different investment banks. Most of the products are linked to indices such as the S&P 500 Index and the Russell 2000 Index. We find that the ARBNs' fair price is approximately 4.5% below the actual issue price. Each of the ARBN's fair price is stable across all four valuation methodologies.

1 Introduction

Structured products are complex debt instruments whose payoffs are linked to the performance of reference stocks, indices, commodity prices, interest rates or exchange rates. Structured products can have very complicated payoff structures, making them difficult for unsophisticated retail investors to value. For example, different products can expose

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investors to all, some, or none of the underlying security's downside risk. Some types of products, such as Principal Protected Absolute Return Barrier Notes (ARBN), expose investors only to the upside potential if certain criteria are met, but not to downside risk. However, all structured products expose investors to the risks of the issuer and the underlying security and many structured products do not guarantee interest. Recently, structured products have become increasingly complex and at the same time have been marketed more to retail investors (Hernández et al., 2007; Henderson and Pearson, 2010).

Several papers have been written on structured products and their valuation. Deng et al. (2009) discusses the different types of structured products and their issue price premia. The paper uses Lehman Brothers as an example of how companies use structured products to finance the firm at below-market interest rates. Deng et al. (2010b) looks at the structured product market as a whole, and presents four generic methods for valuing a wide variety of structured products. Similar to Deng et al. (2009), Deng et al. (2010b) finds that structured products are issued at a premium, possibly indicating a lack of transparent risk disclosures.

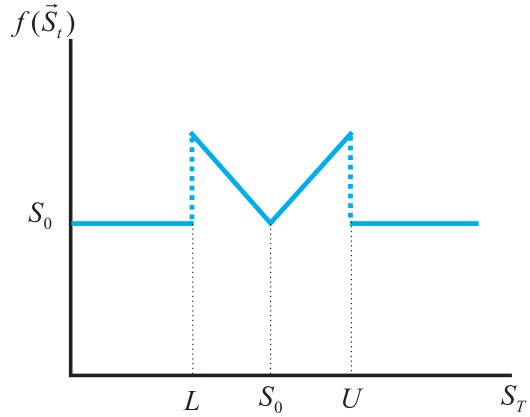
Many papers also explore the characteristics and valuations of specific types of structured products. For example, Hernández et al. (2007), Deng et al. (2010a), and Szymanowska et al. (2009) all explore reverse convertibles. One of the most analyzed structured products are reverse convertibles, that can be converted from a debt instrument into the underlying security at the option of the issuer. All the aforementioned research papers on reverse convertibles indicate a substantial premium on the issue date. Henderson and Pearson (2010) devote the paper to SPARQS, which are closely related to reverse convertibles. Unlike reverse convertibles, SPARQS are callable by the issuer on scheduled call dates. Like reverse convertibles, SPARQS tend to be issued at a premium. Henderson and Pearson (2010) find that *"reasonable estimates of the expecte returns on SPARQS are less than the riskless rate. For the estimates of expected returns on the underlying stocks that seem most reasonable, the average expected return on the SPARQS is actually negative"* (p. 30).

In this paper, we focus on the valuation of Principal Protected Absolute Return Barrier Notes (ARBN). The principal protection feature guarantees the full payback of the note's face value at maturity, as long as the investor holds the note to maturity and the issuer does not default on the note.[1] The interest portion of the ARBN's payoff at maturity is conditional and is linked to the entire return path of the underlying security. If the price of the underlying security remains within a lower and an upper barrier (L and U, respectively) for the entire life of note, the interest included in the payoff at maturity is equal to the absolute value of the underlying security's return. If the price of the

[1] Considering that ARBN are sold as debt instruments, the principal protection is roughly equivalent to saying "we won't default on the debt, unless we default on the debt." See more details in Deng et al. (2009).

underlying security ever crosses the lower or upper barrier, the note does not pay interest. Figure 1 graphs the general payoff structure of ARBN.

Figure 1: Payoff Structure $f(\vec{S}_t)$ of ARBN. L and U are the lower and upper barriers, respectively. S_0 is the initial level of the underlying security. When stock price $\vec{S}_t$, $t \in [0, T]$ remains within barriers, the note receives positive interest. The principal protection feature is indicated by the horizontal line below L and above U.



The pattern of ARBNs' conditional interest payment has similarities to double-barrier options (Li, 1998; Davydov and Linetsky, 2001; Carr et al., 1998). Double-barrier options, which are one of the most popular over-the-counter options (Carr and Crosby, 2008), include both a lower barrier L and an upper barrier U. The function of the barriers depends on whether the option is *knock-in* or *knock-out*. Knock-in double-barrier options cannot be exercised unless the underlying security's price crosses either of the two barriers during the option's contract. In contrast, knock-out double barrier options *lose* their exercise ability if the underlying security's price crosses either of the two barriers.

The simplest double-barrier option is the double-barrier binary option, or double-barrier no-touch option (Carr and Chou, 1997b; Ebenfeld et al., 2002; Hui, 1996). The double-barrier knock-in binary option pays a fixed amount if the price of the underlying security crosses either barrier. If a barrier is not crossed, the option pays nothing. The double-barrier knock-out binary option is similar, but pays the fixed amount *unless* a barrier is crossed. A slightly more complicated option is the double-barrier linear segment option (Li, 1998), which has a payout that varies linearly with the value of the underlying security. Many double-barrier options, including double-barrier call and put options (Kunitomo and Ikeda, 1992), can be formulated as double-barrier linear segment options.

In this paper we present four closed-form approaches to valuing ARBN. The four approaches represent current valuation methodologies on structured products. Although Monte-Carlo simulations may be used to value ARBN, the simulations lack the accuracy of closed-from solutions and require large computational resources. In addition to being faster, the closed-form approach makes it easier to calculate the product's standard comparative statics.[2] The first two approaches we present are based on the decomposition of ARBN payoffs into equivalent portfolios of zero-coupon debt and double-barrier equity options.[3] The first approach decomposes ARBNs into a zero-coupon debt and double-barrier linear segment options. The second approach decomposes ARBNs into a zero-coupon debt and double-barrier call and put options. We use existing closed-form solutions for double-barrier linear segment, call, and put options to value each option separately, and then combine the values to derive the value of the ARBNs.

The third approach values the ARBNs by numerical integration of the payoff function. Because the ARBNs' payoff structure depends on both the underlying security's price during the contract's term and the underlying security's final price (i.e., the payoff structure is path-dependent), it cannot be directly integrated. Instead, we first transform the ARBN's payoff function into an equivalent portfolio of path-independent payoff functions (i.e., payoff functions that depend only on the underlying security's final price) using the techniques introduced in Carr and Chou (1997b). Numerical integration also simplifies the calculation of ARBN comparative statics such as delta, which is used to hedge the underlying securities.

The fourth approach is the Partial Differential Equation (PDE) valuation approach (Black and Scholes, 1973; Wilmott et al., 1994). PDE is a versatile tool with many modeling capabilities, including many difficult-to-model features such as American options, convertible bonds, etc. The Black-Scholes equation is a classical model to the price of the structured product. Combining the payoff terms in ABRNs such as principal protection and absolute return, we solve the PDE in closed form using Fourier series expansions. As a matter of fact, PDE is relatively difficult to solve, thus many formulated PDE can only be solved using numerical methods such as finite difference methods. For ARBNs, we obtain a neat closed-form solution.

We apply the four valuation approaches to 214 ARBNs issued by six investment banks: Deutsche Bank, Goldman Saches, HSBC, Lehman Brothers, Morgan Stanley, and UBS. We model the price of each ARBN at issuance, and find that the products have a 4.5%

[2]The standard comparative statics are commonly called 'Greeks', and include delta. Delta measures the sensitivity of an option's value to the price of the underlying security.

[3]See Deng et al. (2010b) for a discussion of the benefits and drawbacks of the decomposition method. A portfolio replication approach includes dynamic replication and static replication (Derman et al., 1995). A dynamic replication states that a replicated by a portfolio of stocks and bonds with varying weights along time. While in this paper we refer to the static replication.

issue premium on average relative to the price we model. The implied yield for these products is generally lower than the issuer's corporate yield, and in some cases is even lower than the risk-free rate. We analyze and summarize the actual returns of all the matured ARBNs.

The paper is organized as follows. In Section 2 we derive the closed-form valuation equations for each of the four approaches. In Section 3 we use the four approaches to value our sample of ARBNs. In Section 4 we recap the contents and conclude the paper.

2 Valuation of the Notes

Section 2.1 introduces the valuation assumptions which we maintain for all four valuation techniques. Section 2.2 and Section 2.3 present the two decomposition approaches, Section 2.4 explains the payoff transformation methodology, and Section 2.5 discusses the partial differential equation valuation method.

2.1 Assumptions

We use the assumptions included in the Black-Scholes model for option valuation. We assume the underlying security price S_t follows a geometric Brownian Motion

$$dS_t = \mu S dt + \sigma S dZ_t, \tag{1}$$

where, under the risk-neutral measure, μ is a constant drift defined as

$$\mu = r - q.$$

r is the risk free rate and q is the dividend yield of the underlying security.[4] σ is the volatility of the process and Z_t is a standard Brownian motion. At maturity ($t = T$), the underlying security's price, conditional on the underlying security's initial price S_0, is log-normally distributed

$$S_T | \, S_0 \sim \text{Log-}\mathcal{N}\left(\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right) T, \sigma\sqrt{T}\right). \tag{2}$$

At maturity, ARBN returns the face value to the investors. In addition, if the underlying security's price remains within the barriers $[L, U]$ for all $t \in [0, T]$, the ARBN pays a return equal to the absolute value of the underlying security's return. For simplicity,

[4]We assume both the risk free rate and the dividend yield are constant over time in the model.

we introduce two variables denoting the first hitting time of the stock price on barriers: τ_L and τ_U . They are defined as

$$\tau_L = \inf\{t | \, S_t = L\}$$

and

$$\tau_U = \inf\{t | \, S_t = U\}.$$

Thus, the ARBN pays a return beyond the face value of the note only if $\min(\tau_L, \tau_U) > T$. The ARBN payoff function, $f(\vec{S}_t)$, is written as

$$f(\vec{S}_t) \quad = \quad \begin{cases} S_0 + |\frac{S_T - S_0}{S_0}|, & \text{when } \min(\tau_L, \tau_U) > T, \\ S_0, & \text{otherwise.} \end{cases} \tag{3}$$

2.2 Decomposition Using Linear Segment Options

In the decomposition approach, the structured product's payoff is broken down into an equivalent portfolio of simple bond instruments, options contracts, forward contracts, and swaps. All of the securities in the equivalent portfolio have closed-form, relatively straightforward valuations. By combining the values of each security in the equivalent portfolio, we derive value of the more complex structured product.

In this section we decompose ARBN into a zero-coupon bond and two knock-out, double-barrier linear segment options (DBLS).[5] A double-barrier linear segment option pays $a + bS_T$ when the final equity price S_T falls inside the interval $[X_1, X_2]$, and 0 otherwise. The payoff of a knock-out DBLS, shown graphically in Figure 2, is

$$f_{DBLS}(\vec{S}_t, a, b, X_1, X_2) \quad = \quad \begin{cases} a + bS_T, & \text{when } \min(\tau_L, \tau_U) > T \text{ and } S_T \in [X_1, X_2], \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

A DBLS without barriers, called a linear segment option (LS), is a generalized option from which other options are derived as special cases. For example, a binary option is a LS option with parameters $(a, b, X_1, X_2) = (1, 0, X_1, X_2)$; a call option is a LS option with parameters $(a, b, X_1, X_2) = (-S_0, 1, S_0, \infty)$; and a put option is a LS option with parameters $(a, b, X_1, X_2) = (S_0, -1, 0, S_0)$.

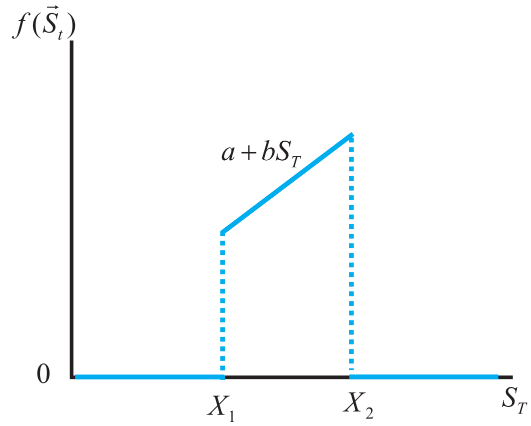
A simple derivation shows that the payoff of ARBN is decomposed as

$$f_{ARBN}(\vec{S}_t) = S_0 + f_{DBLS}(\vec{S}_t, -S_0, 1, S_0, U) + f_{DBLS}(\vec{S}_t, S_0, -1, L, S_0). \tag{5}$$

The equation indicates that the ARBN payoff is equivalent to a three-security portfolio: a) 1 share of zero-coupon debt with a face value of S_0, b) 1 share of a knock-out DBLS

[5] See Li (1998) for a more detailed discussion of double-barrier linear segment options.

Figure 2: Payoff mapping function of DBLS with parameters (a, b, X_1, X_2), when the barriers L and U are not breached.



with parameters $(a, b, X_1, X_2) = (-S_0, 1, S_0, U)$, and c) 1 share of a knock-out DBLS with parameters $(a, b, X_1, X_2) = (S_0, -1, L, S_0)$. The ARBN's fair value on issue date is derived as:

$$V_{ARBN}(\vec{S}_t) = e^{-(r+\bar{C})T} S_0 + V_{DBLS}(\vec{S}_t, -S_0, 1, S_0, U) + V_{DBLS}(\vec{S}_t, S_0, -1, L, S_0). \quad (6)$$

Consistent with Deng et al. (2010b), we include the credit default swap (CDS) spread $\bar{C}$ of the issuer in the discount factor. Including the CDS adjusts the valuation for the counterparty risk faced by the investor.[6]

Li (1998) provides the following valuation formula for knock-out DBLS

$$\begin{aligned} V_{DBLS}(\vec{S}_t, a, b, X_1, X_2) &= \sum_{n=-\infty}^{\infty} \left[V_{LS}\left(S_0 \left(\frac{U}{L}\right)^{2n}, a, b, X_1, X_2 \right) - \right. \\ &\left. V_{LS}\left(\frac{U^2}{S_0} \left(\frac{U}{L}\right)^{2n}, a, b, X_1, X_2 \right) \left(\frac{U}{S_0}\right)^{\frac{2\lambda}{\sigma^2}} \right] \left(\frac{U}{L}\right)^{\frac{2n\lambda}{\sigma^2}}. \end{aligned}$$

λ is a constant equal to $r + q - \sigma^2/2$. The standard linear segment option value V_{LS} is given as

$$V_{LS}(a, b, X_1, X_2) = a e^{-(r+\bar{C})T} \left[N\left(d_2^{(X_1)}\right) - N\left(d_2^{(X_2)}\right) \right] +$$

[6]See (Hull, 2008) on how to adjust derivative price with counterparty credit risk. Credit risk does affect the prices of structured products. This is reflected by the fact that prices of Bear Stearns structured products dropped significantly in March 2008, and bounced back when JP Morgan announced its acquisition.

$$bS_0 e^{-(q+\bar{C})T}\left[N\left(d_1^{(X_1)}\right) - N\left(d_1^{(X_2)}\right)\right],$$

where

$$d_1^{(X)} = \frac{\log(S_0/X) + (r - q + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}, \quad d_2^{(X)} = d_1^{(X)} - \sigma\sqrt{T}.$$

2.3 Decomposition Using Call and Put Options

Another intuitive way to decompose ARBN is using knock-out double-barrier call options (DBC) and knock-out double-barrier put options (DBP). The payoff of a knock-out DBC is

$$f_{DBC}(\vec{S}_t, K) = \begin{cases} \max(S_T - K, 0), & \text{when } \min(\tau_L, \tau_U) > T, \\ 0, & \text{otherwise.} \end{cases}$$

Similarly, the payoff of a knock-out DBP is

$$f_{DBP}(\vec{S}_t, K) = \begin{cases} \max(K - S_T, 0), & \text{when } \min(\tau_L, \tau_U) > T, \\ 0, & \text{otherwise.} \end{cases}$$

In both cases, K is the option's strike price. Note that the value of the DBC, $V_{DBC}(\vec{S}_t, K)$, is zero when $K \geq U$ because the option is underwater until after the knock-out criterion is satisfied. The same is true for DBP with strike prices below the lower barrier L

$$V_{DBP}(\vec{S}_t, K) = 0, \text{when } K \leq L.$$

A simple derivation provides an equivalent portfolio for ARBN payoffs $f_{ARBN}(\vec{S}_t)$ that consists of zero-coupon debt and knock-out double-barrier call and put options

$$f_{ARBN}(\vec{S}_t) = S_0 + f_{DBC}(\vec{S}_t, S_0) + f_{DBP}(\vec{S}_t, S_0). \tag{7}$$

Thus, the value of any ARBN is:

$$V_{ARBN}(\vec{S}_t) = e^{-(r+\bar{C})T} S_0 + V_{DBC}(\vec{S}_t, S_0) + V_{DBP}(\vec{S}_t, S_0). \tag{8}$$

The valuation of DBC and DBP have been well documented in many papers (Kunitomo and Ikeda, 1992). To be consistent among the decomposition methods presented in this paper, we rewrite the DBC and DBP in terms of DBLS. The value of the DBC in Eqn. (8) is:

$$\begin{aligned} V_{DBC}(\vec{S}_t, S_0) &= V_{DBLS}(\vec{S}_t, -S_0, 1, S_0, \infty) \\ &= V_{DBLS}(\vec{S}_t, -S_0, 1, S_0, U). \end{aligned}$$

The second equation comes from the fact that there is no positive return when the underlying stock's price exceeds the upper barrier U. The BDP component in Eqn. (8) is calculated as

$$\begin{aligned} V_{DBP}(\vec{S}_t, S_0) &= V_{DBLS}(\vec{S}_t, S_0, -1, 0, S_0) \\ &= V_{DBLS}(\vec{S}_t, S_0, -1, L, S_0), \end{aligned}$$

again replacing the non-binding constraint $X_1 = 0$ with the binding constraint $X_1 = L$. The value of the note coincides with the value derived in Eqn. (6).

2.4 Transforming the Payoff Function

ARBNs have path-dependent payoff functions in the form $f_{ARBN}(\vec{S}_t)$. In other words, the ARBN's payoff depends on a vector, $\vec{S}_t$, of the underlying security's prices for all $t \in [0, T]$. The path-dependent property of the payoff function is an intriguing point to handle, since it introduces discrete jumps in the valuation. If we transform the payoff into an equivalent portfolio of path-independent payoff functions, $\hat{f}_{ARBN}(S_T)$, which depends only on S_T, we can value ARBNs by integration. The new payoff function $\hat{f}_{ARBN}(S_T)$ is often referred as an *adjusted payoff function.* Given the adjusted payoff function, a structured product's fair value is simply the discounted value of the expected payoff

$$V_{ARBN}(\vec{S}_t) = \hat{V}_{ARBN}(S_T) = e^{-(r+\bar{C})T} \int_0^\infty \hat{f}_{ARBN}(S_T) pdf(S_T) dS_T. \tag{9}$$

The transformation technique is based on Carr and Chou (1997a,b)' reflecting theory on rewriting path-independent equivalent payoff functions. This theory is helpful in handling path-dependent option valuations, such as barrier options, rolling options and lookback options. The following lemma summarizes the relevant portions of Carr and Chou (1997b).

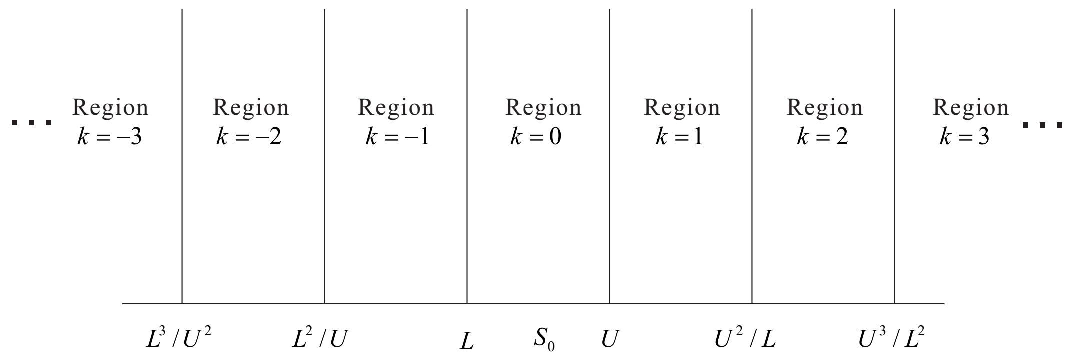
Lemma 1 *In a double-barrier option, let $f(S_T)$ be the final payoff function when neither barrier has been breached. The adjusted payoff is defined piecewise on the regions which form the price range $(0, \infty)$*

$$\hat{f}(S_T) = \begin{cases} f(S_T), & \text{in regions } k = 0; \\ \underbrace{R_D \circ R_U \circ R_D}_{k \text{ operators}} \cdots (f(S_T)), & \text{in regions } k = -1, 2, -3 \ldots \\ \underbrace{R_U \circ R_D \circ R_U}_{k \text{ operators}} \cdots (f(S_T)), & \text{in regions } k = 1, -2, 3 \ldots \end{cases} \tag{10}$$

where the two reflection operators are defined as

$$R_D(f(S_T)) = -\left(\frac{S_T}{L}\right)^p f(L^2/S_T) \text{ and } R_U(f(S_T)) = -\left(\frac{S_T}{U}\right)^p f(U^2/S_T)$$

Figure 3: Dividing the regions $(0, \infty)$



and p is a constant defined as $p = 1 - 2\frac{\mu}{\sigma^2}$.

Notice that because the reflection theory in Lemma 1 intuitively divides the price range $(0, \infty)$ into infinitely many intervals, the closed-form valuation of double-barrier options is an infinite sum which has a form similar to the result derived in Section 2.2.

Let $\tilde{f}_{ARBN}(S_T)$ be the final payoff function of an ARBN where neither barrier was breached. The form of the payoff function is

$$\tilde{f}_{ARN}(S_T) = S_0 + |S_T - S_0|, \quad \forall\, S_T \in [L, U].$$

Applying the Lemma, the adjusted payoff for an ARBN is calculated as (written in the form of $\hat{f}_{ARBN(k)}(S_T)$ defined on the kth region)

$$\hat{f}_{ARBN(k)}(S_T) = \begin{cases} -\left(\frac{S_T}{U}\right)^p \left(\frac{L}{U}\right)^{jp} \left|\frac{U^{2j+2}}{L^{2j} S_T} - S_0\right| + S_0, & \text{in regions } k = 2j + 1; \\ \left(\frac{U}{L}\right)^{jp} \left|\frac{L^{2j}}{U^{2j}} S_T - S_0\right| + S_0, & \text{in regions } k = 2j. \end{cases} \tag{11}$$

For a given set of parameters, we plot the adjusted payoff function in Figure 4.

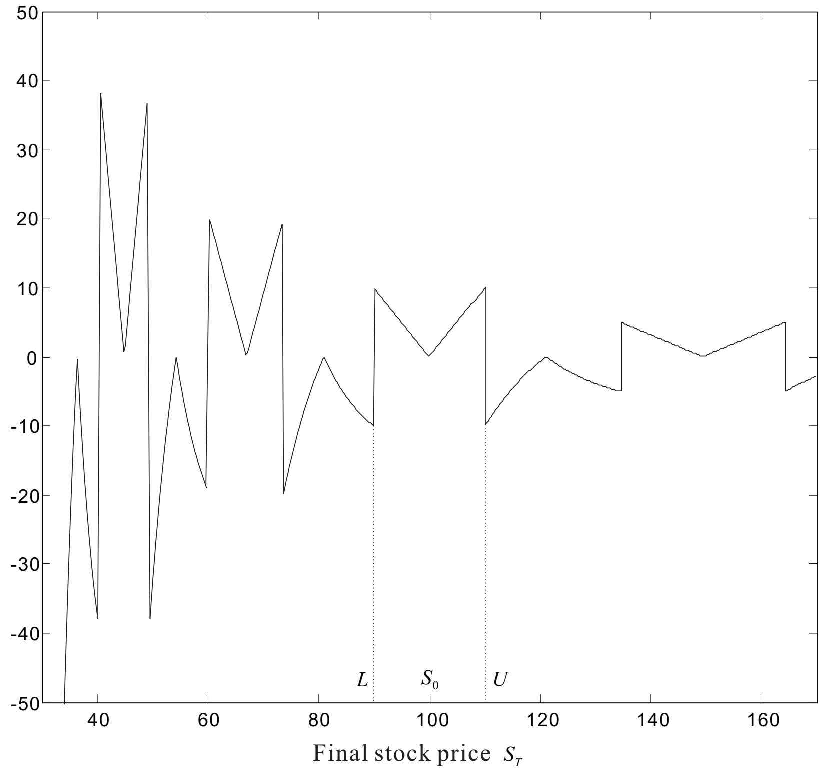
By plugging the adjusted payoff in Eqn. (9), the valuation for an ARBN is

$$\hat{V}_{ARBN}(S_T) = \sum_{k=-\infty}^{\infty} \hat{V}_{ARBN(k)}(S_T), \tag{12}$$

where each component

$$\hat{V}_{ARBN(k)}(S_T) = e^{-(r+\bar{C})T} S_0 + e^{-(r+\bar{C})T} \;. \tag{13}$$

Figure 4: Plot of the adjusted payoff function for $\hat{f}_{ARBN}(S_T) - S_0$. (Subtracting S_0 is to help visualization of variations.) Parameters used are $r = 0.05, \sigma = 0.15, q = 0.005, T = 1, S_0 = 100, L = 90, U = 110, \bar{C} = 0.003$. The function is defined piecewisely in regions. Notice that the function $\hat{f}(x)$ is same as $\tilde{f}_{ARN}(x)$ in the region [L, U].



$$\begin{cases} \begin{array}{l} -\frac{L^{jp}}{U^{(j+1)p}} \Big(S_0 e^{\frac{w^2p^2}{2}+mp} \big(N(d(\frac{U^{2j+1}}{L^{2j}}) - wp) + \\ N(d(\frac{U^{2j+2}}{L^{2j+1}}) - wp) - 2N(d(\frac{U^{2j+2}}{L^{2j}S_0}) - wp)\big) - \\ Ae^{\frac{w^2(p-1)^2}{2}+m(p-1)} \big(N(d(\frac{U^{2j+1}}{L^{2j}}) - wp + w) + \\ N(d(\frac{U^{2j+2}}{L^{2j+1}}) - wp + w) - 2N(d(\frac{U^{2j+2}}{L^{2j}S_0}) - wp + w)\big)\Big) \end{array}, & \text{in regions } k = 2j+1; \\ \begin{array}{l} \frac{U^{jp}}{L^{jp}} \Big(S_0 \big(2N(d(\frac{U^{2j}S_0}{L^{2j}})) - N(d(\frac{U^{2j}}{L^{2j-1}})) - N(d(\frac{U^{2j+1}}{L^{2j}}))\big) - \\ Be^{\frac{w^2}{2}+m} \big(2N(d(\frac{U^{2j}S_0}{L^{2j}}) - w) - N(d(\frac{U^{2j}}{L^{2j-1}}) - w) \\ -N(d(\frac{U^{k+1}}{L^{k}}) - w)\big)\Big) \end{array}, & \text{in regions } k = 2j; \end{cases}$$

where the constants are $m = \ln(S_0) + (r - q - \sigma^2/2)T$, and $w = \sigma\sqrt{T}$. The constants

m and w are the mean and standard deviation for $\log(S_T)$ respectively. The function $N(\cdot)$ represents the standard normal cumulative distribution function and the function $d(X) = \frac{\log X - m}{w}$.

The path-independent function $\hat{f}_{ARBN}(S_T)$ is also useful in calculating the Greeks of ARBNs. For example, to calculate delta, referring to the formula in Deng et al. (2010b), the delta of the product at time t is

$$delta_t = e^{-(r+\bar{C})T} \int_{-\infty}^{\infty} \hat{f}'_{ARBN}(e^{Ww+m}) \frac{e^{Ww+m}}{S_t} pdf(W) dW$$

where W is a standard normal variable.

2.5 PDE Modeling

In the PDE approach, the value dynamics of structured products is modeled as a two-dimensional function $V(S,t)$, where S is the stock price and t is the valuation time. Similar to the modeling of plain vanilla options, the value satisfies the classical Black-Scholes equation (Black and Scholes, 1973; Wilmott et al., 1994).

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - (r+\bar{C})V = 0. \tag{14}$$

The fair value for ARBN at issuance is the valuation function at time 0 with initial price S_0:

$$V_{ARBN}(\vec{S}_t) = V(S_0, 0).$$

See Deng et al. (2010b) for more details on using PDE to model structured product. We apply an approach similar to Hui (1996) to solve the Black-Scholes equation using Fourier series.

Let us consider the boundary conditions first. When the underlying security's price crosses either barrier, the ARBN pays S_0 at maturity. Therefore, the boundary conditions are

$$V(L,t) = S_0 e^{-(r+\bar{C})(T-t)}, \quad V(U,t) = S_0 e^{-(r+\bar{C})(T-t)}.$$

If the underlying security's price never breaches either barrier, the final conditional satisfies the payoff at maturity

$$V(S,T) = S_0 + |S - S_0|, \quad \text{for } L < S < H.$$

We adapt the transformation of variables presented in Wilmott et al. (1994), a process termed 'dimensionless', to simplify the equation. The new variables introduced (x, τ) satisfy the following equations:

$$S = Le^x, \quad t = T - \frac{2\tau}{\sigma^2}, \quad V(S,t) = Le^{\alpha x + \beta\tau} u(x,\tau) + S_0\, e^{-(r+\bar{C})(T-t)},$$

where the constants are

$$k_1 = \frac{2(r-q)}{\sigma^2}, \quad \alpha = -\frac{1}{2}(k_1 - 1), \quad \beta = -\frac{1}{4}(k_1-1)^2 - \frac{2(r+\bar{C})}{\sigma^2}.$$

The Black-Scholes equation is then be simplified as a heat equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \qquad \text{for } 0 < x < \log\frac{U}{L}, \quad \tau > 0. \tag{15}$$

The corresponding boundary conditions and initial condition[7] become

$$u(0,\tau) = 0, \quad u\left(\log\frac{U}{L}, \tau\right) = 0, \quad u(x,0) = \left|e^x - \frac{S_0}{L}\right| e^{-\alpha x}, \text{ for } 0 < x < \log\frac{U}{L}.$$

This results a standard diffusion equation with boundary conditions. Suppose the solution to the equation is of the form

$$u(x,\tau) = \sum_{n=1}^{\infty} D_n e^{-\frac{n^2\pi^2\tau}{\left(\log\frac{U}{L}\right)^2}} \sin\left(\frac{n\pi x}{\log\frac{U}{L}}\right),$$

where D_n is a parameter to be determined by the initial condition. We may solve D_n in a Fourier series form. Defining new constants $\hat{U} = \log\frac{U}{L}$ and $\hat{P} = \log\frac{S_0}{L}$, we derive

$$\begin{aligned}
& D_n \\
= \; & \frac{2}{\hat{U}} \int_0^{\hat{U}} \left|e^x - \frac{S_0}{L}\right| e^{-\alpha x} \sin\left(\frac{n\pi x}{\hat{U}}\right) dx \\
= \; & \frac{2}{\hat{U}} \left(\int_{\hat{P}}^{\hat{U}} \left(e^x - \frac{S_0}{L}\right) e^{-\alpha x} \sin\left(\frac{n\pi x}{\hat{U}}\right) dx - \int_0^{\hat{P}} \left(e^x - \frac{S_0}{L}\right) e^{-\alpha x} \sin\left(\frac{n\pi x}{\hat{U}}\right) dx \right) \\
= \; & \frac{-2e^{\hat{P}}}{n^2\pi^2 + \alpha^2\hat{U}^2} \left(-n\pi\left(1 + (-1)^n e^{-\alpha\hat{U}}\right) + 2n\pi e^{-\hat{P}\alpha} \cos\left(\frac{\hat{P}n\pi}{\hat{U}}\right) + 2\alpha\hat{U} e^{-\hat{P}\alpha} \sin\left(\frac{\hat{P}n\pi}{\hat{U}}\right)\right) \\
& + \frac{2}{\alpha^2\hat{U}^2 - 2\hat{U}^2\alpha + \hat{U}^2 + n^2\pi^2} \left(-n\pi + 2n\pi e^{-(\alpha-1)\hat{P}} \cos\left(\frac{\hat{P}n\pi}{\hat{U}}\right) \right. \\
& \left. + 2\hat{U} e^{-(\alpha-1)\hat{P}} \sin\left(\frac{\hat{P}n\pi}{\hat{U}}\right)(\alpha - 1) - e^{-(\alpha-1)\hat{U}} n\pi(-1)^n \right).
\end{aligned}$$

By plugging in the above solution D_n and reversing the transformation of the state variables, we derive the ARBN's value at any time t

$$V(S,t) = Le^{\alpha x + \beta\tau} \sum_{n=1}^{\infty} D_n e^{-\frac{n^2\pi^2\tau}{\hat{U}^2}} \sin\left(\frac{n\pi x}{\hat{U}}\right) + S_0 \, e^{-(r+\bar{C})(T-t)}, \tag{16}$$

[7]Original final condition changes to initial condition after changing of variables.

using the reversion of transformations

$$\tau = \frac{\sigma^2(T-t)}{2} \text{ and } x = \log\left(\frac{S}{L}\right).$$

The convergence speed for this function is fast. To calculate V at time 0, typically requires $n = 10$.

3 Real World ARBN

We collect data on 279 ARBNs issued by six investment banks from 2006 and 2009. The six investment banks–Deutsche Bank, Goldman Sachs, HSBC, Lehman Brothers, Morgan Stanley and UBS–market their ARBNs under slightly different names. For example, Goldman Saches calls them 'Absolute Return Trigger Notes.' Besides the standar type, there are also other variations in the structure of ARBNs. For example, *autocallable* ARBNs pay the principal back when the barriers are breached rather than at maturity, and *buffered* ARBNs provide a buffer against losses but do not protect 100% of the note's principal.

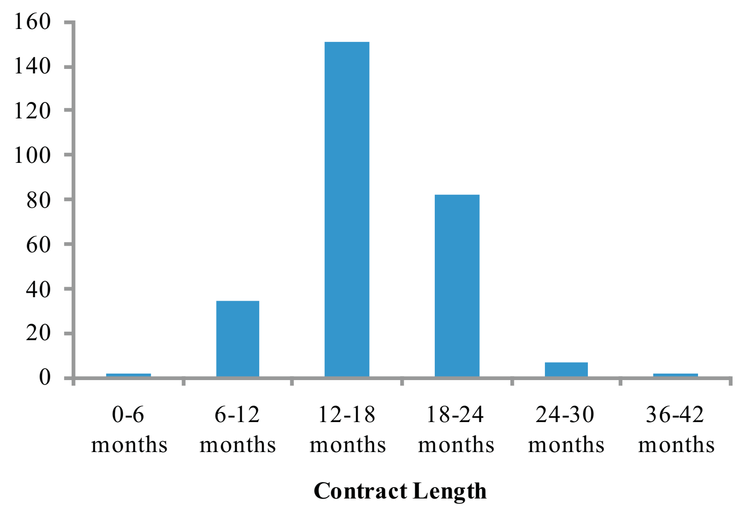
The 279 ARBNs have an aggregate face value of $4.2 billion, and are generally linked to indices. The primary indices used are the S&P 500 Index (63% of issues) and the Russell 2000 Index (13%). Others underlying securities include the Nasdaq 100 Index, EFTs, and exchange rates. The narrow distribution of underlying securities in line with Henderson and Pearson (2010)'s conclusion that issuers prefer using well-known underlying securities.

Table 1: Distribution of ARBN Underlying Securities

Issuers	S&P 500 Index	Russell 2000 Index	Other Indices	Non-index	Total
Deutsche Bank AG	60	15	8	19	102
Goldman Sachs	26		1	7	34
HSBC	3	4	1	2	10
Lehman Brothers	10	3	4	2	19
Morgan Stanley	30		12	5	47
UBS	47	15	3	2	67
Total	176	37	29	37	279

Generally speaking, ARBNs are short-term investments. Figure 5 charts the maturities of the ARBNs in our sample. The maturities range from 6 months to 3 years, but most maturities are between 12 and 18 months. The underwriting charges on these product range from 1% to 2%. Underwriting fees include a fee paid to investment companies who market the product and commissions paid to the investment advisor who sold the product to the investor. In our sample, UBS is frequently used as the marketing agent. The issuer must charge more than the underwriting fees plus the market value of the product in order to make a profit.We restrict our sample to standard ARBNs for which we can collect the necessary information, including the underlying security's implied volatility. Our final sample set contains 214 ARBNs.

Figure 5: Contract Length Distribution



We apply the four closed-form approaches to calculate the fair value of the notes at issuance. All of the approaches yield exactly the same fair price. We also perform Monte Carlo simulations to simulate the underlying security's prices, which yield a value very close to the other four methods. On average, the fair value of an ARBN in our sample is 95.5% of the product's principal, meaning the note is issued at a 4.5% premium. In the following table, we present the ARBNs' fair values and implied yields by investment banks. The implied yield is defined as the interest rate that makes the fair value equal to the issue price (at par). Morgan Stanley tends to have a lower average price at issuance than the other issuers. This is probably due to Morgan Stanley's high credit default swap spread, which is incorporated in the valuation model.

Figures 6(a) to 6(c) plot the implied yield of each ARBN against the 1-year LIBOR

Table 2: Fair Valuation of the ARBNs

Issuer	Fair Price at Issue Time	Average Implied Yield
Deutsche Bank AG	95.94%	1.31%
Goldman Sachs	95.69%	1.00%
HSBC	96.69%	1.47%
Lehman Brothers	95.06%	1.49%
Morgan Stanley	91.92%	1.11%
UBS	96.86%	1.32%
Total	95.58%	1.32%

rate and the issuer's 1-year bond yield.[8] As the figures show, all of the ARBNs' implied yields are lower than the corresponding corporate yields, and many are even lower than the risk-free rate. Similar to Deng et al. (2009), we find that Lehman's structured products generally have implied yields below the 1-year LIBOR rate. This indicates that Lehman used structured products to debt-finance its operations at sub-market rates, especially when the company's credit quality decreased sharply in 2007 and 2008.

173 of the 214 ARBNs have matured. Of the 173 issues, 11 defaulted (all Lehman's), 119 breached a barrier and returned the face value to investors, and 43 paid investors a positive return. Considering all 162 (173-11) of the matured ARBNs, the average return on the notes was 3.5%. For the 43 issues that paid a positive return, the average return was 13.4%.

Table 3: Actual Returns for the ARBNs

Issuers	Total Matured Products	Positive Returns	Pay Principal
Deutsche Bank AG	68	15	53
Goldman Sachs	22	7	15
HSBC	8	2	6
Lehman Brothers	11 (defaulted)	3	8
Morgan Stanley	16	5	11
UBS	48	14	34
Total	173	46	127

[8]For simplicity, the corporate bond yield is equivalent to sum of the LIBOR rate and the issuer's CDS.

4 Conclusion

In this paper we present four closed-form solutions to Principal Protected Absolute Return Barrier Notes. The first two methods are based on decomposing the complex structured product into simple components, which are zero-coupon debts and double-barrier type options. The decomposition methods may use existing results on component options.

The third method transforms the path-dependent structured product to an equivalent path-independent product, using the reflecting theory. The transformed product is easy to value by integrating the path-independent payoff function with the distribution function of the ending stock price.

The fourth method models ARBN by partial differential equations. The payoff is modeled as final conditions and the barrier crossing criteria are modeled as boundary conditions. We solve the PDE by transforming it into a standard diffusion equation and with Fourier series expansions.

We demonstrate each valuation method on our sample of 214 ARBNs. We find that the ARBNs in our sample are issued at a 4.5% premium. This premium is lower than the premia on European reverse convertible products presented in Henderson and Pearson (2010) but is close to the premia on U.S. dollar-denominated reverse convertibles discussed in Hernández et al. (2007). We further find that the risk brought about by underlying securities with higher implied volatility is offset by a higher barrier.

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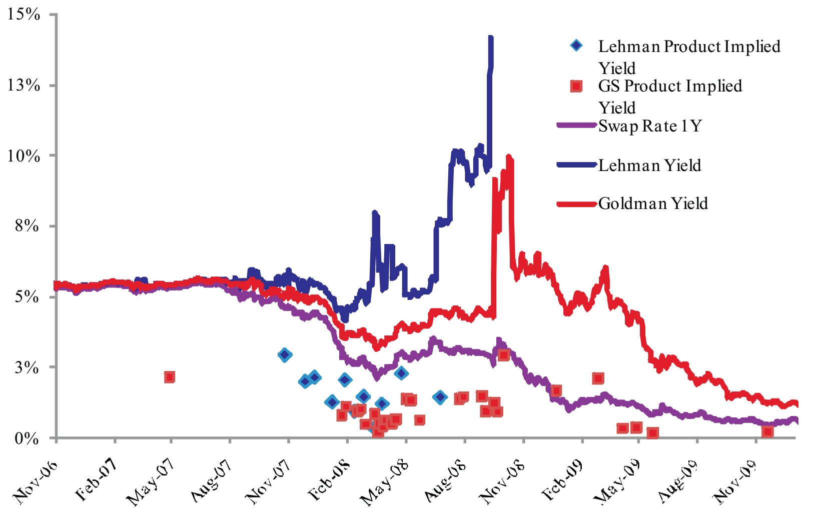
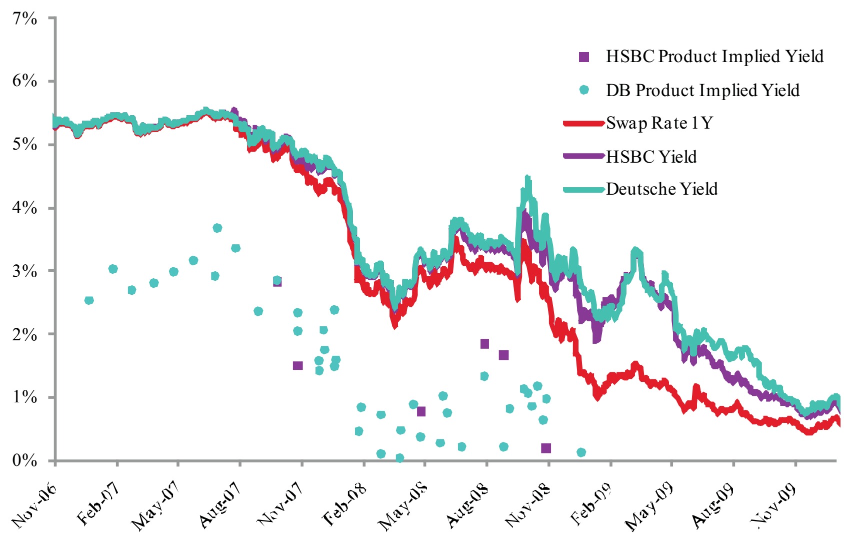
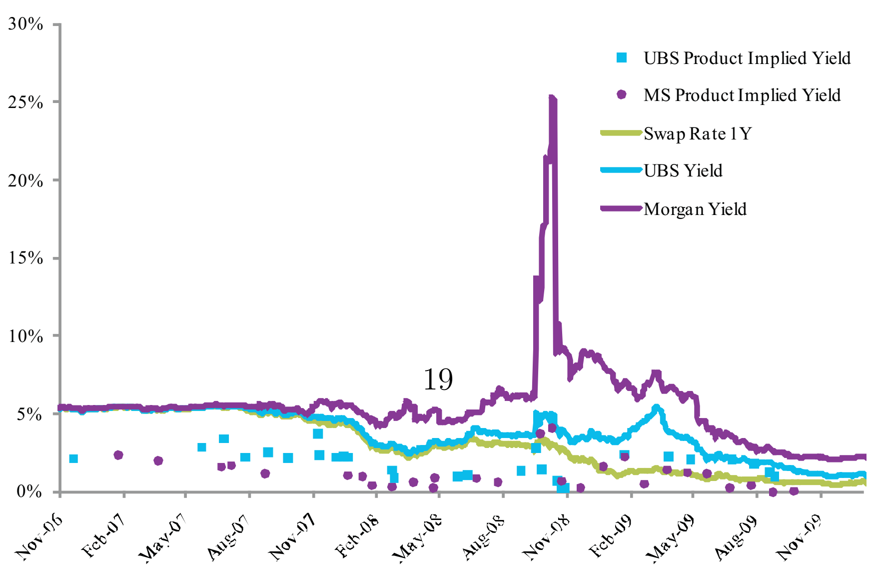


(a) Lehman Brothers and Goldman Sachs



(b) HSBC and Deutsche Bank



(c) UBS and Morgan Stanley